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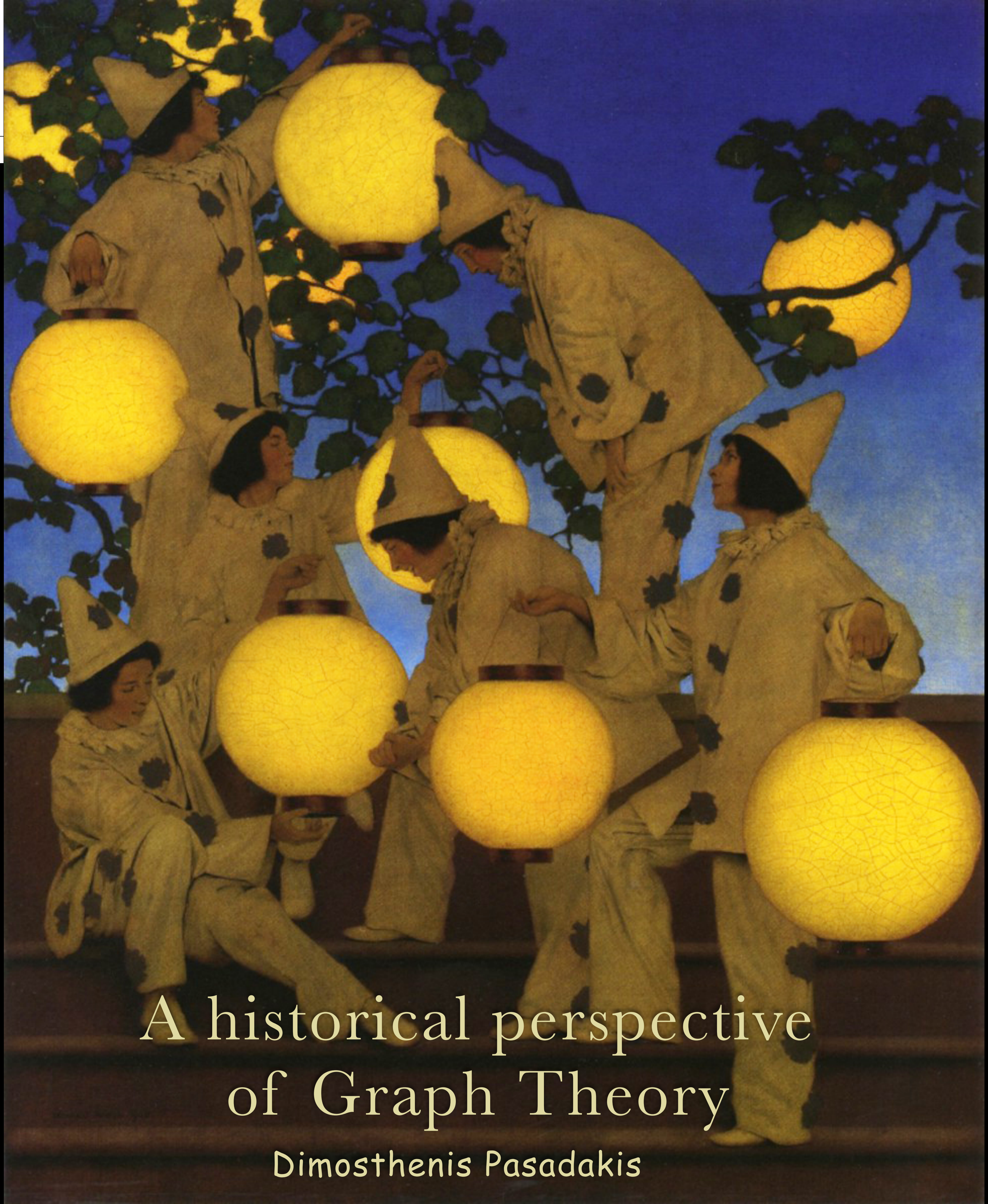

A historical perspective of Graph Theory

Dimosthenis Pasadakis

Image Credits: Maxfield Parrish - The Lantern Bearers, 1908

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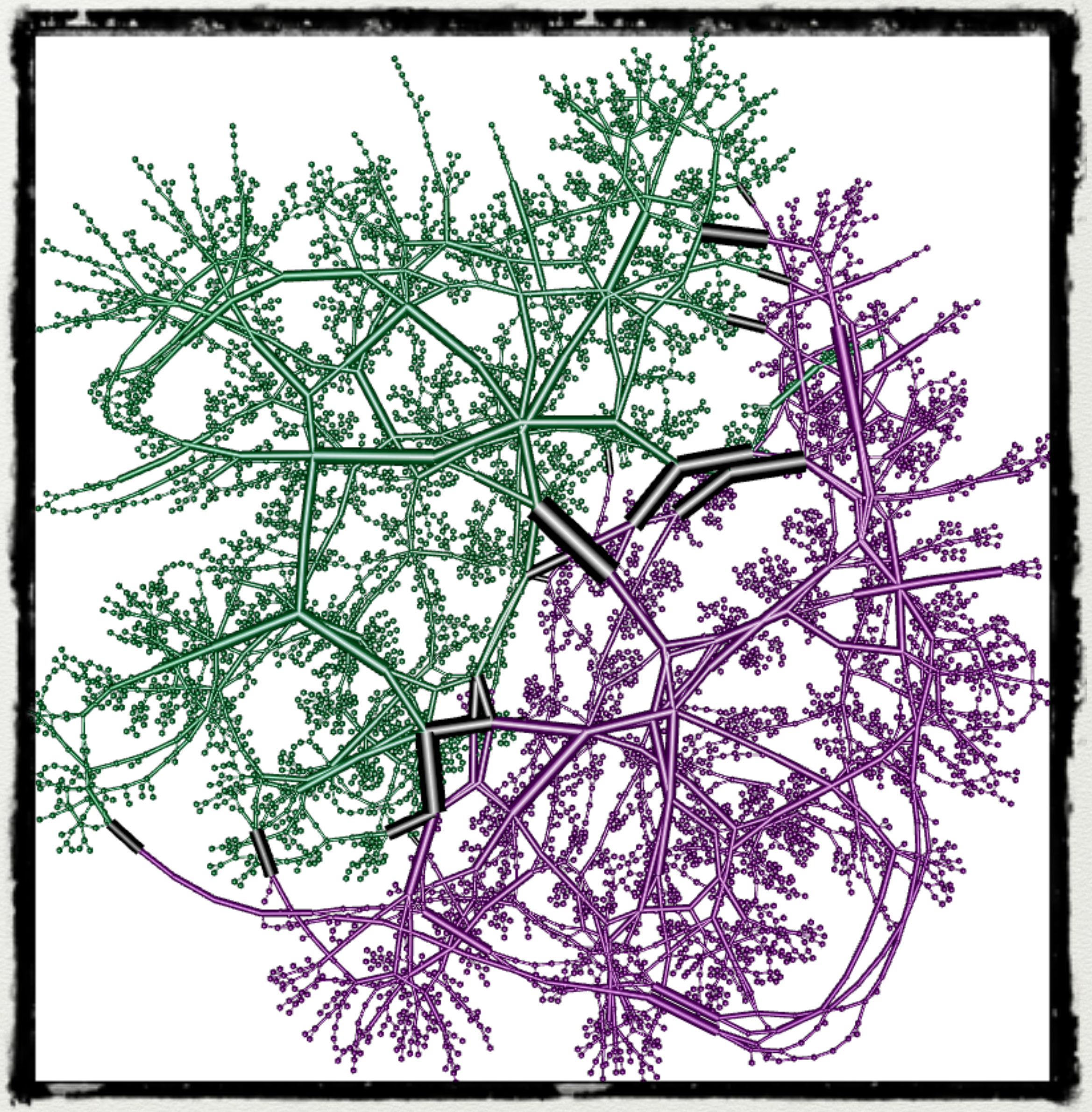
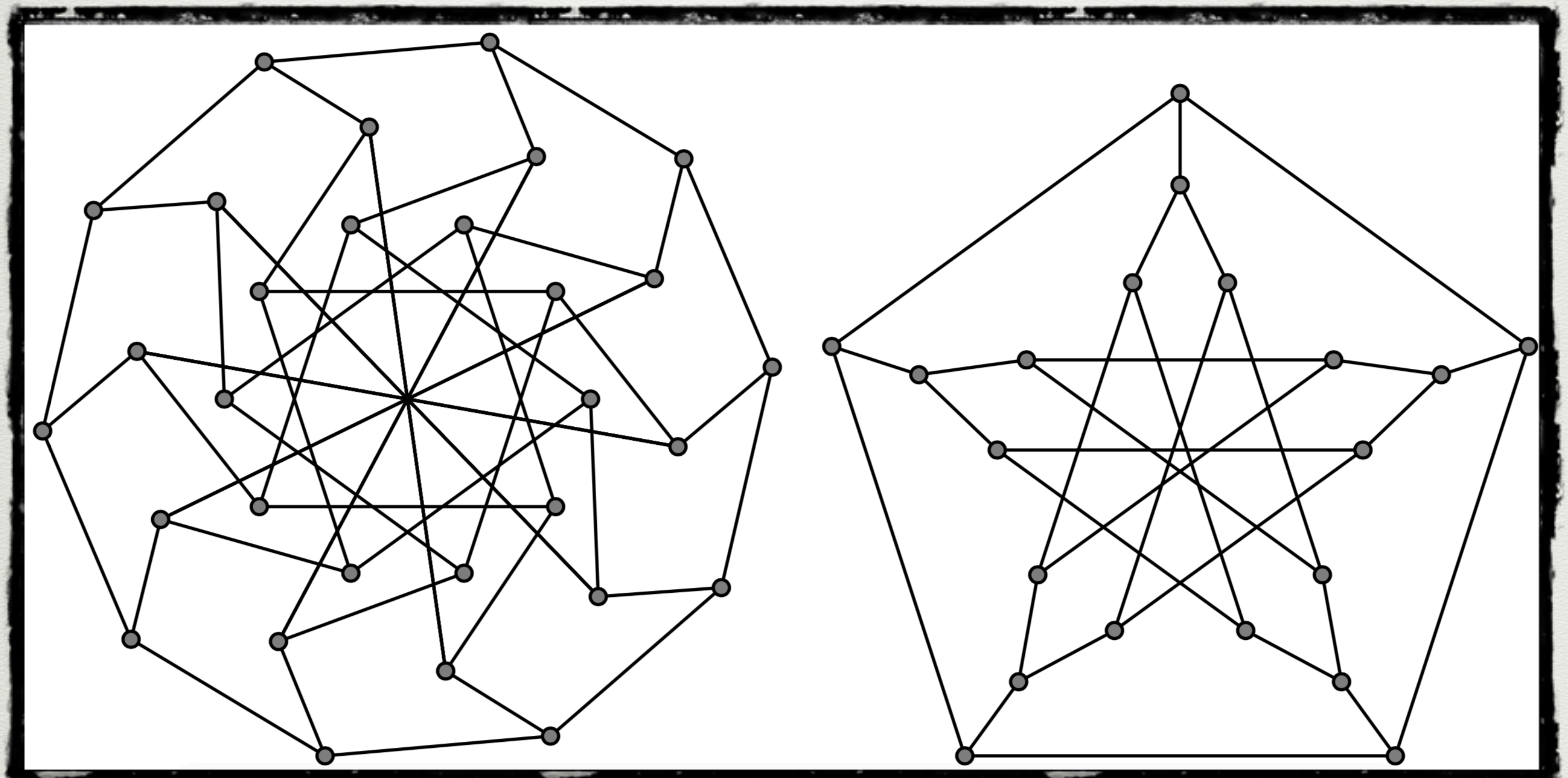


The AC power network of France, partitioned in 2 pieces.

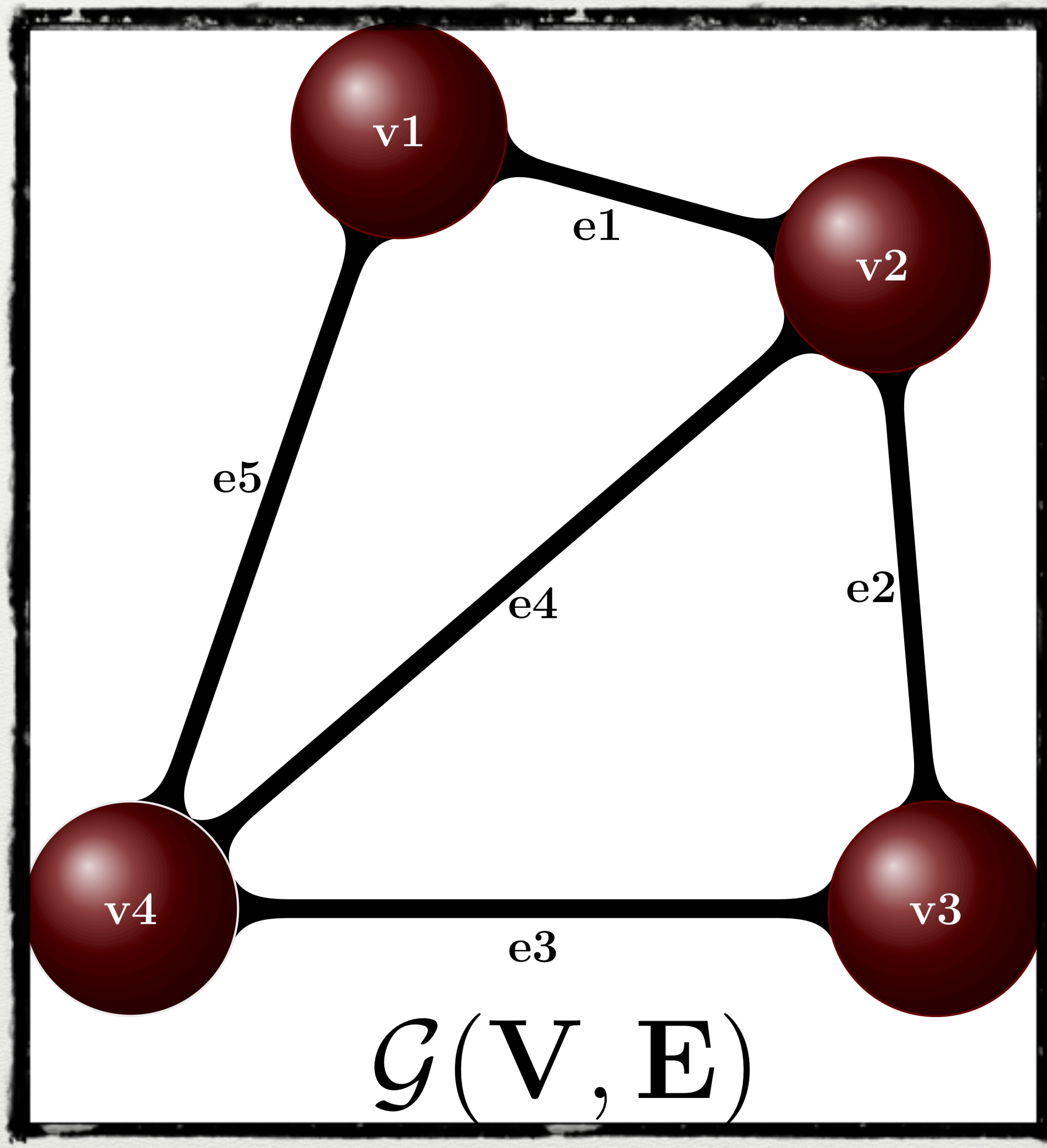


Left: smallest cubic graph of girth 8.
Right: largest cubic graph of diameter 5.

1. Definition of a Graph

Definition of a Graph

- A mathematical entity that encompasses the notion of objects (nodes, vertices), and the connection between them (edges).
- Edge: describes the way the nodes are related.
- Used for exploratory data analysis, with application ranging from statistics, computer science, biology, psychology etc.



Representation of a Graph

Graphs can be represented by matrices that encode their information. The vital information that needs to be transmitted from such a matrix is:

1. which node is connected with whom, and

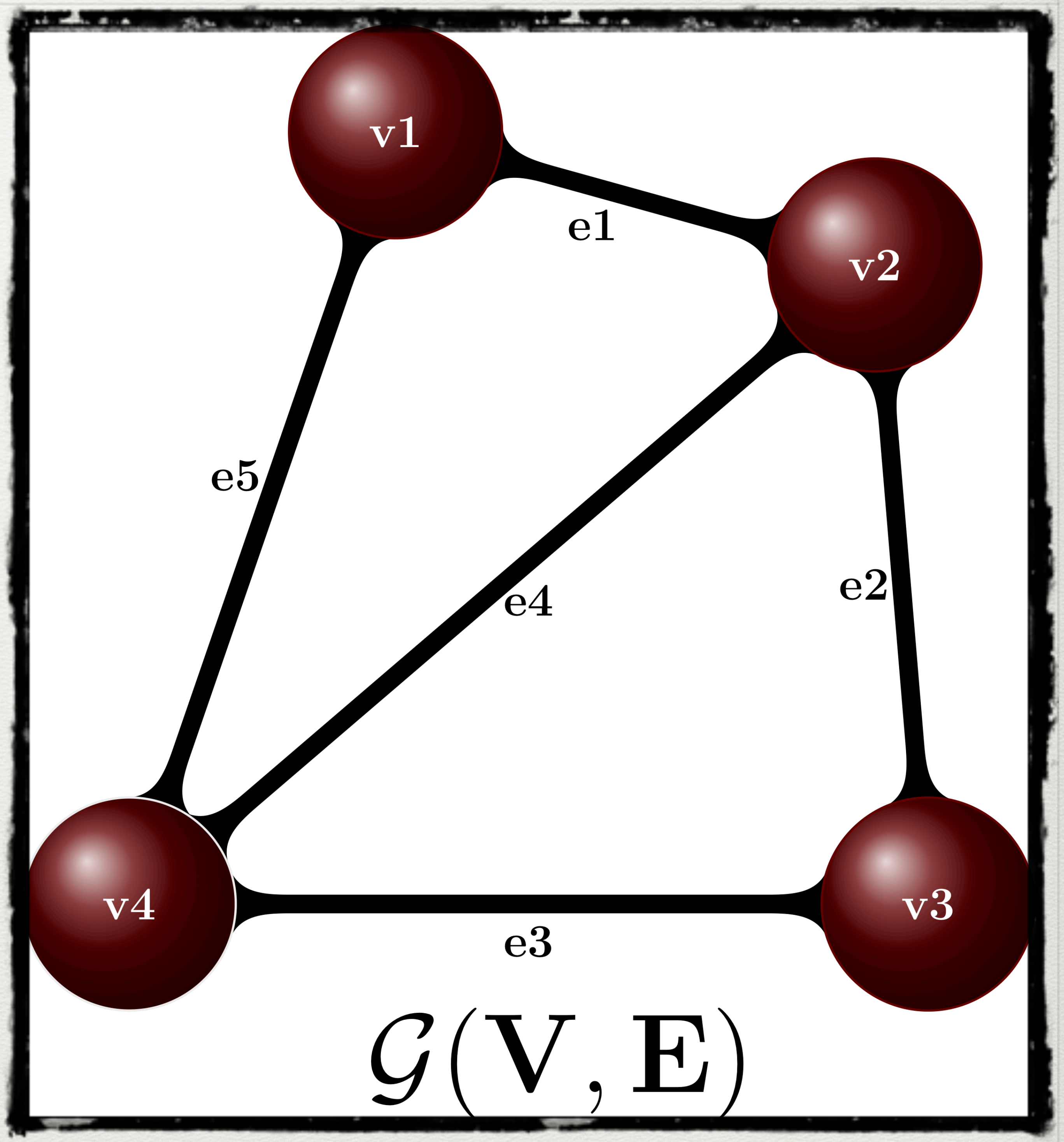
2. the importance (weight) of this connection.

Let n : number of vertices,

m : number of edges .

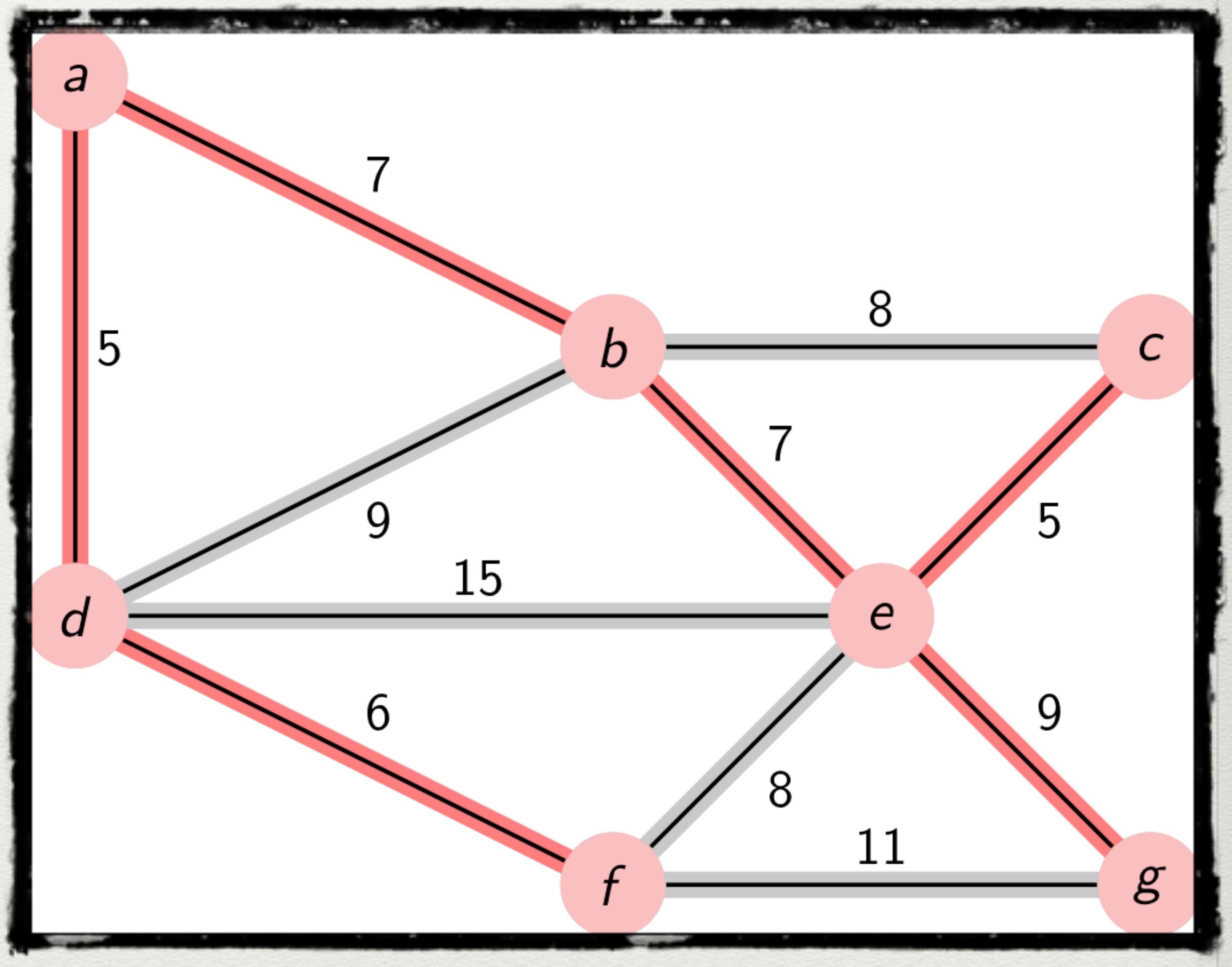
The Graph Laplacian: $\mathbf{L} \in \mathbb{R}^{n \times n}$

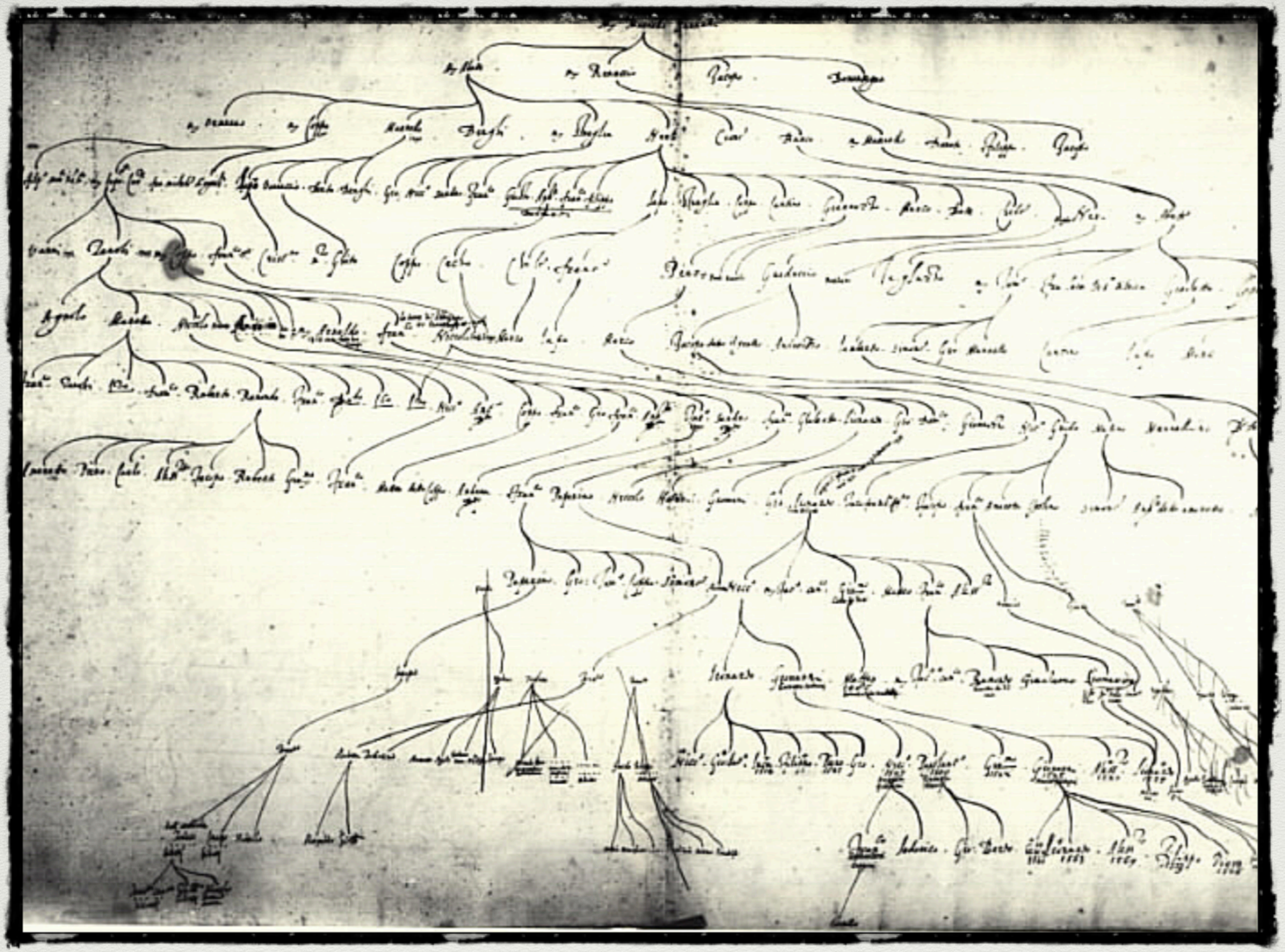
$$\mathbf{L} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$



Trees as Graphs

- Tree: an undirected graph in which any two vertices are connected by exactly one path.
- Spanning tree: a subgraph that is a tree which includes all of the vertices of G, with minimum possible number of edges.
- Minimum spanning tree: all of the above + minimum total edge weight.

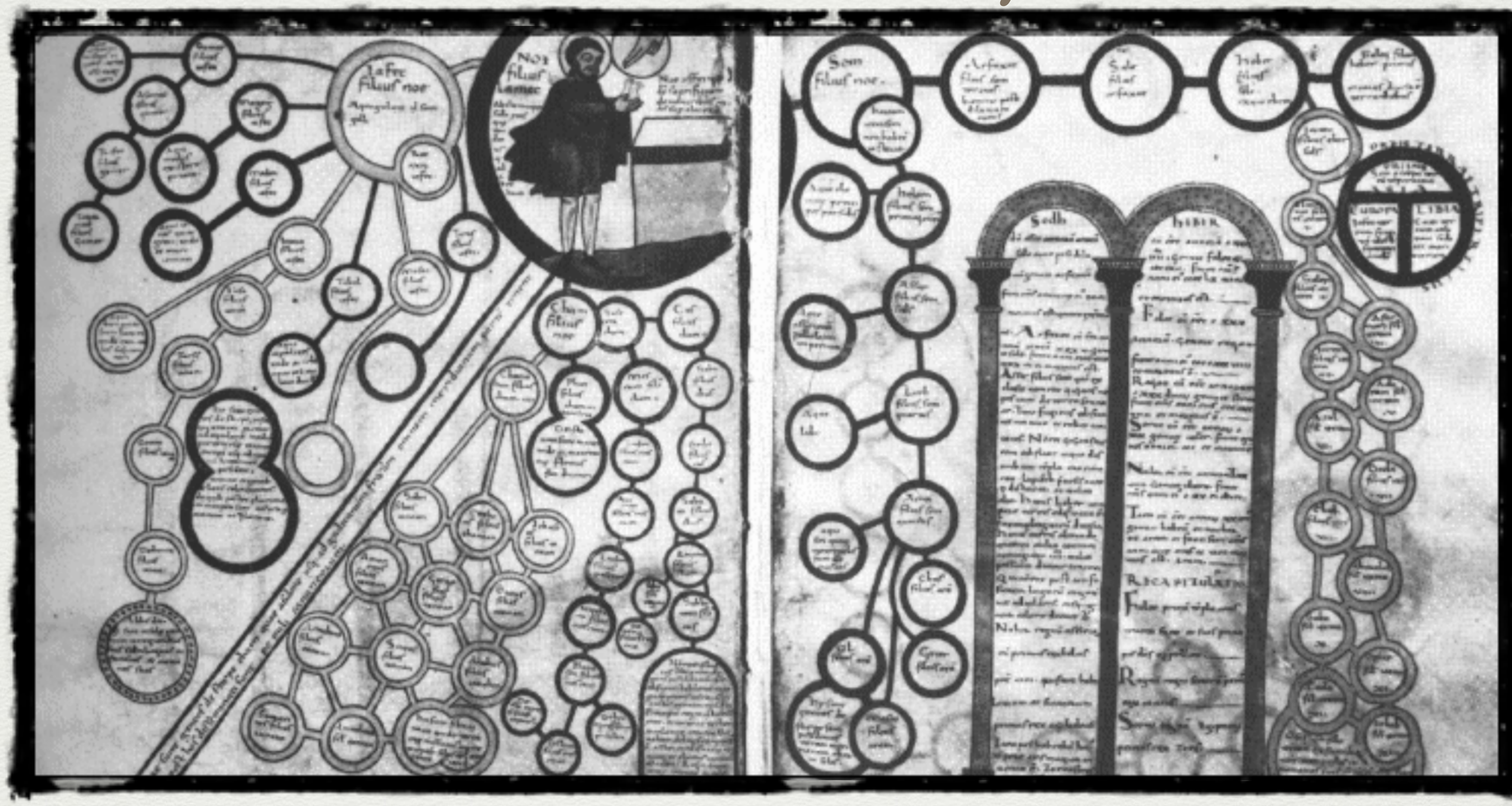




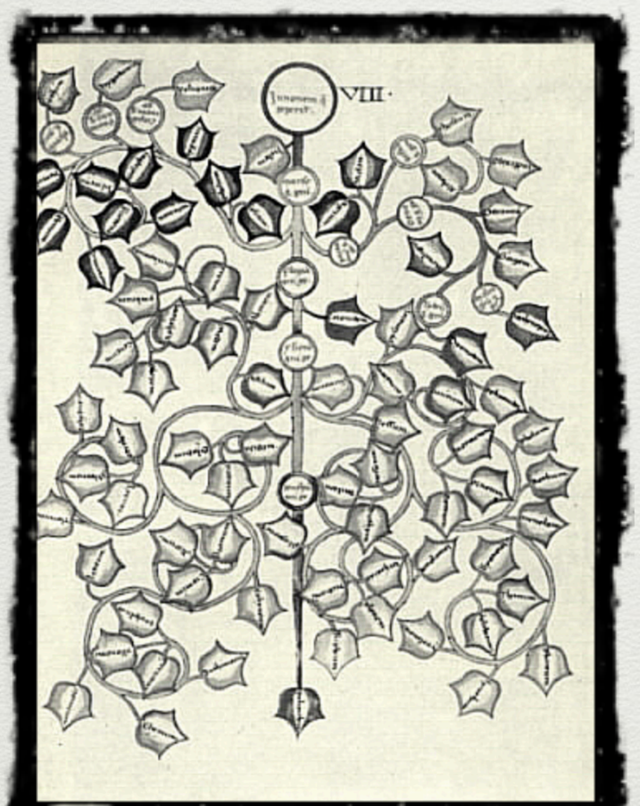
Family-tree of the Mannelli family, Florence, 15th century [1].

2. *Early Graph Drawing*

Medieval Family Trees



Noah’s descendants, 11th century [1]

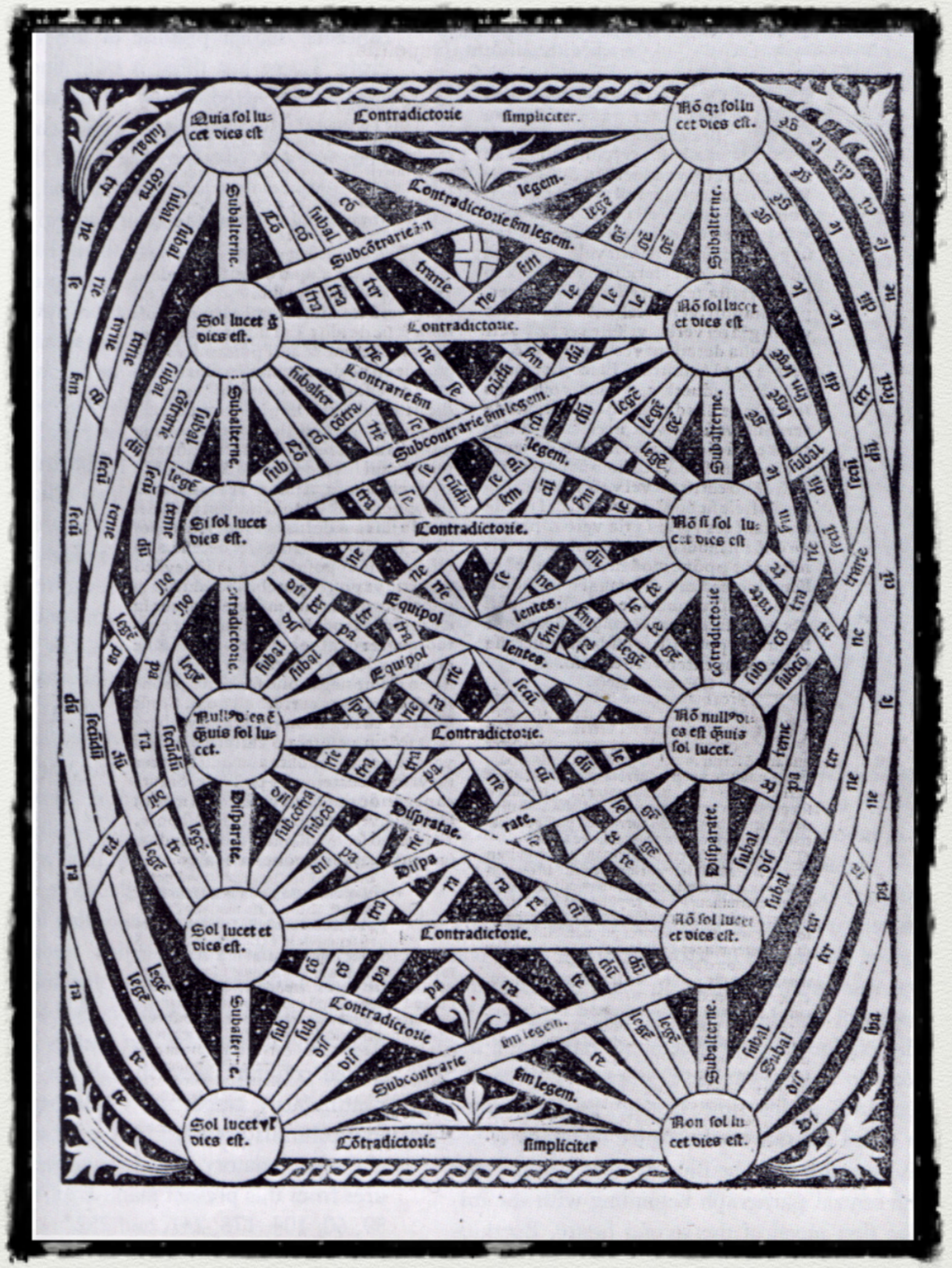


Religious genealogy, 11th century [3]

The Saxon dynasty, 12th century [1]

Other Medieval Graphs

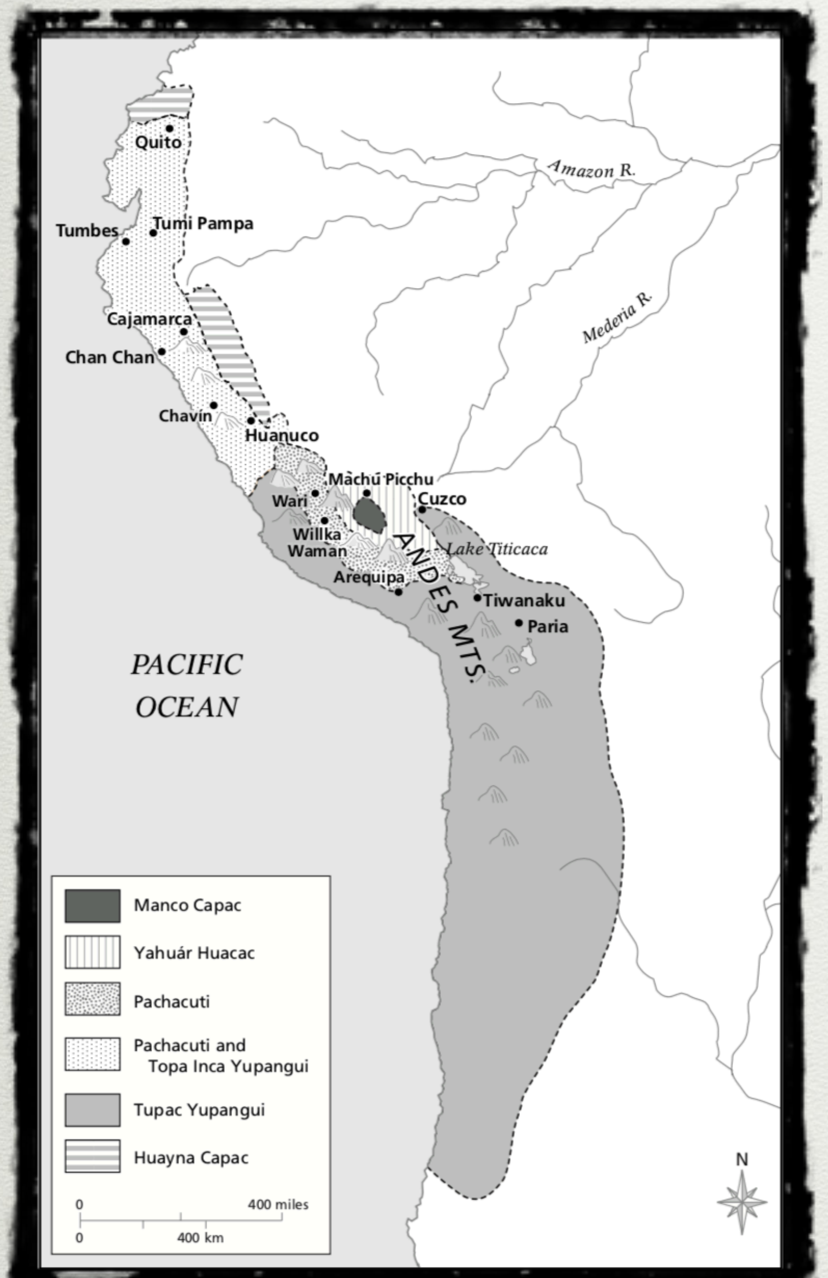
A "Tree of Virtues" (Arbor Virtutum), 14th century [1]



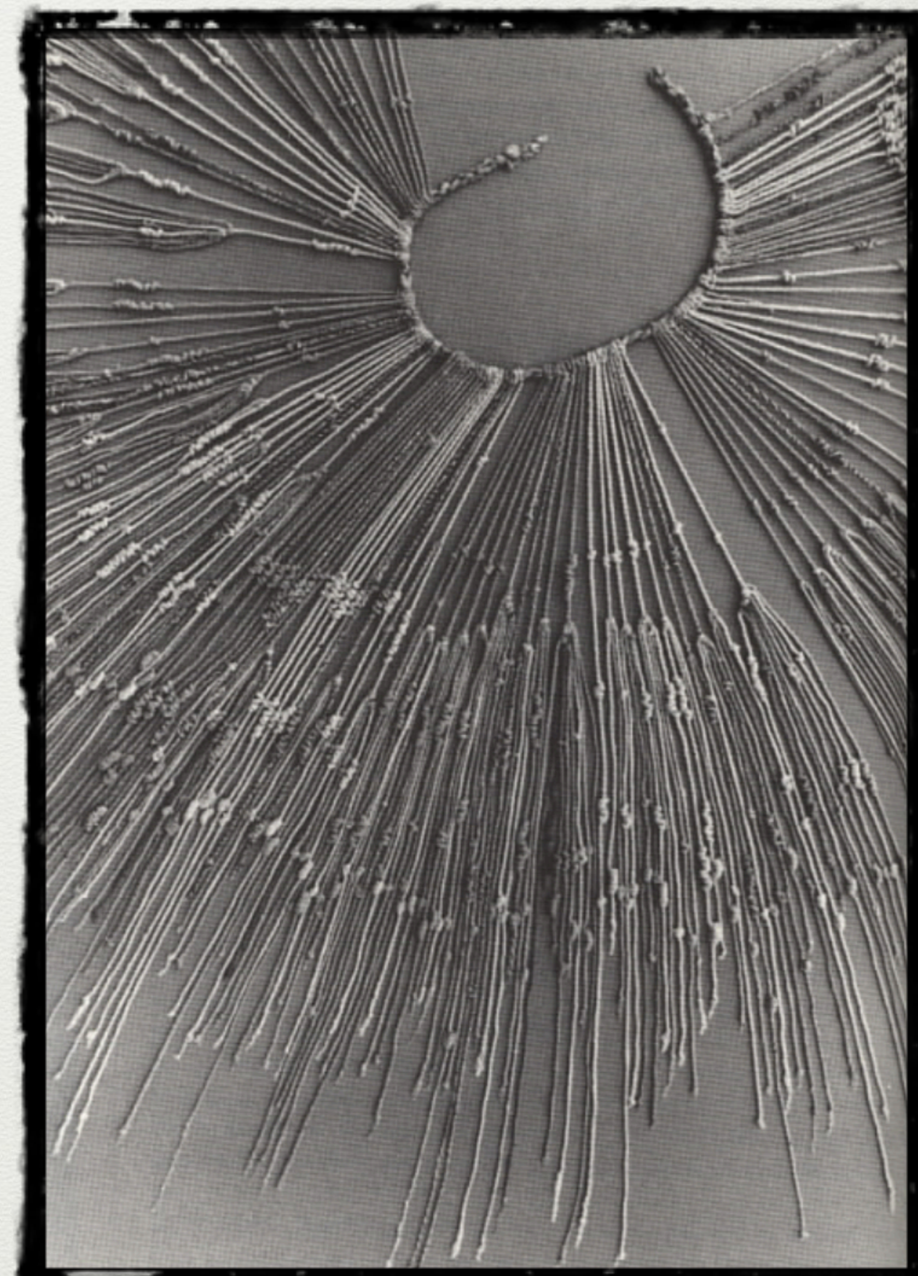
A square of opposition. It is a symmetric graph with labeled nodes and edges. 16th century[1]

Graphs in the New World

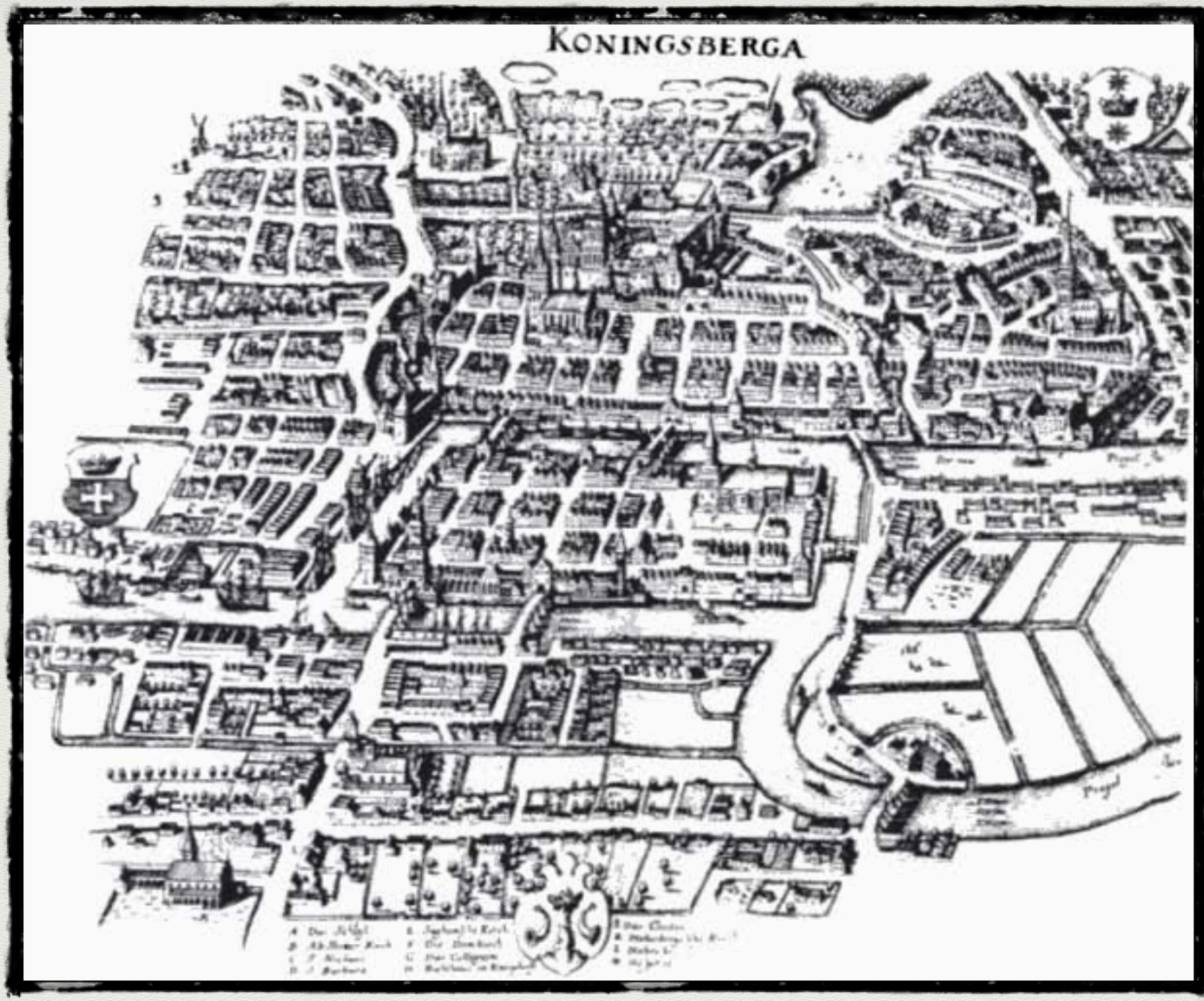
The first Inca ruler, 18th century Peruvian oil painting [5]



The Incas empire, with its vassal states [5]



A quipu in the collection of the Museo National de Antrhopologia y Arquelogía, Lima, Peru [4]



The old city of Königsberg,
[6].

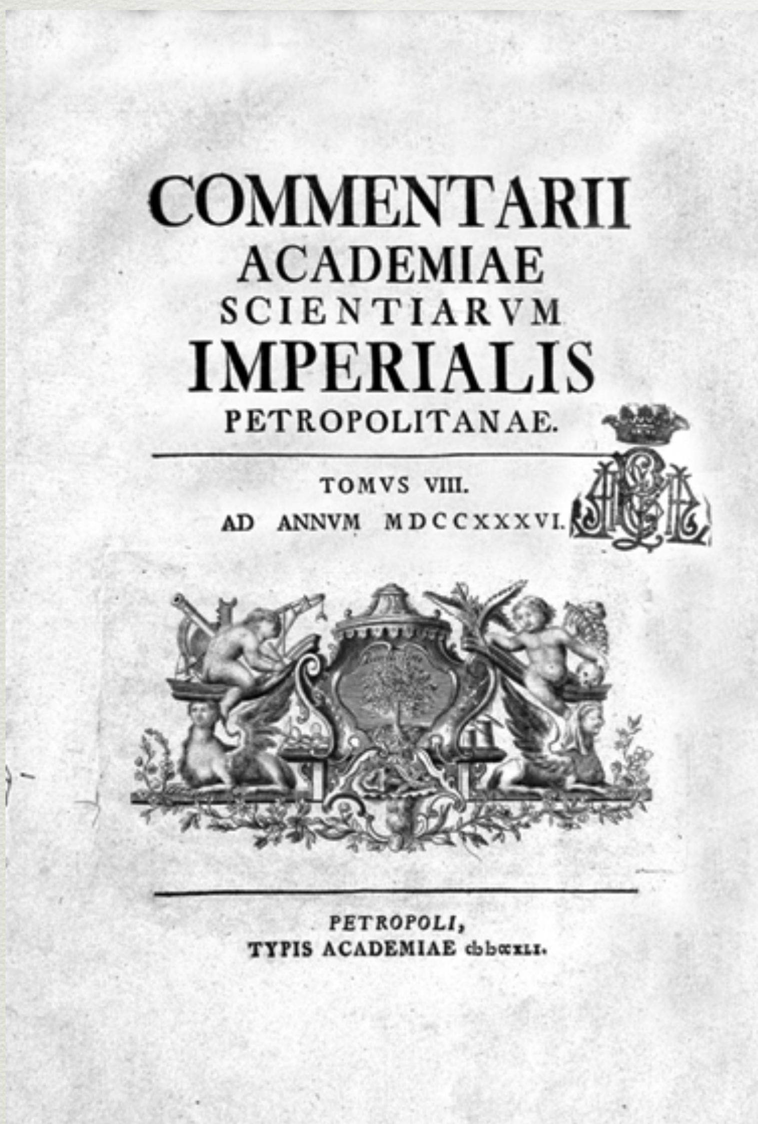
3. Euler and Graph Theory

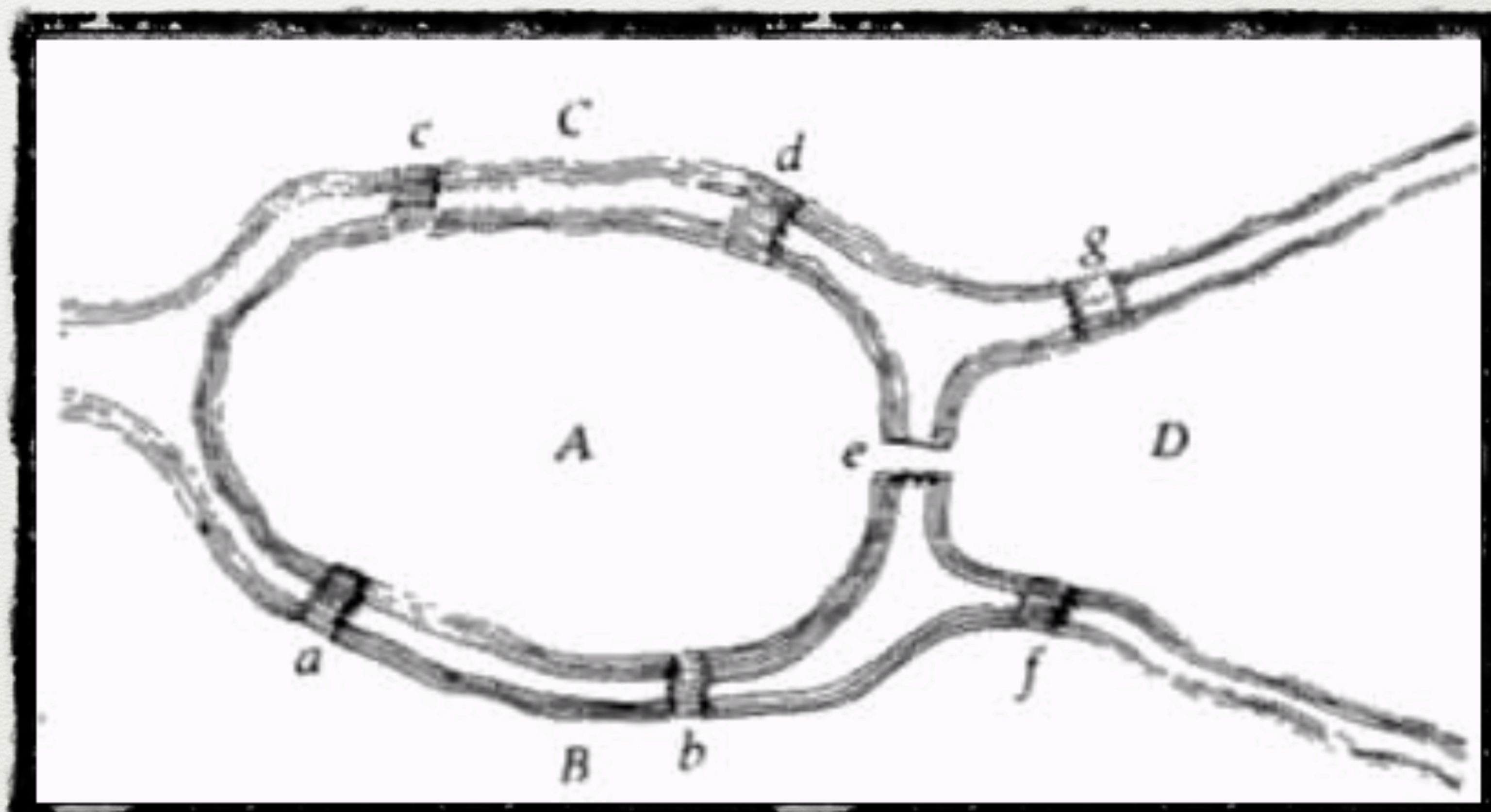
L. EULER

SOLUTIO PROBLEMATIS AD GEOMETRIAM SITUS PERTINENTIS

[The solution of a problem relating to the geometry of position]

Commentarii Academiae Scientiarum Imperialis Petropolitanae **8** (1736), 128–140.
(Based on a talk presented to the Academy on 26 August 1735.)





The problem, which I am told is widely known, is as follows: in Königsberg in Prussia, there is an island ***A,*** *called the Kneiphof; the river which surrounds it is divided into two branches, and these branches are crossed by seven bridges,* ***a***, ***b***, ***c***, ***d***, ***e***, ***f*** *and* ***g***. *Concerning these bridges, it was asked whether anyone could arrange a route in such a way that he would cross each bridge once and only once.* [6]

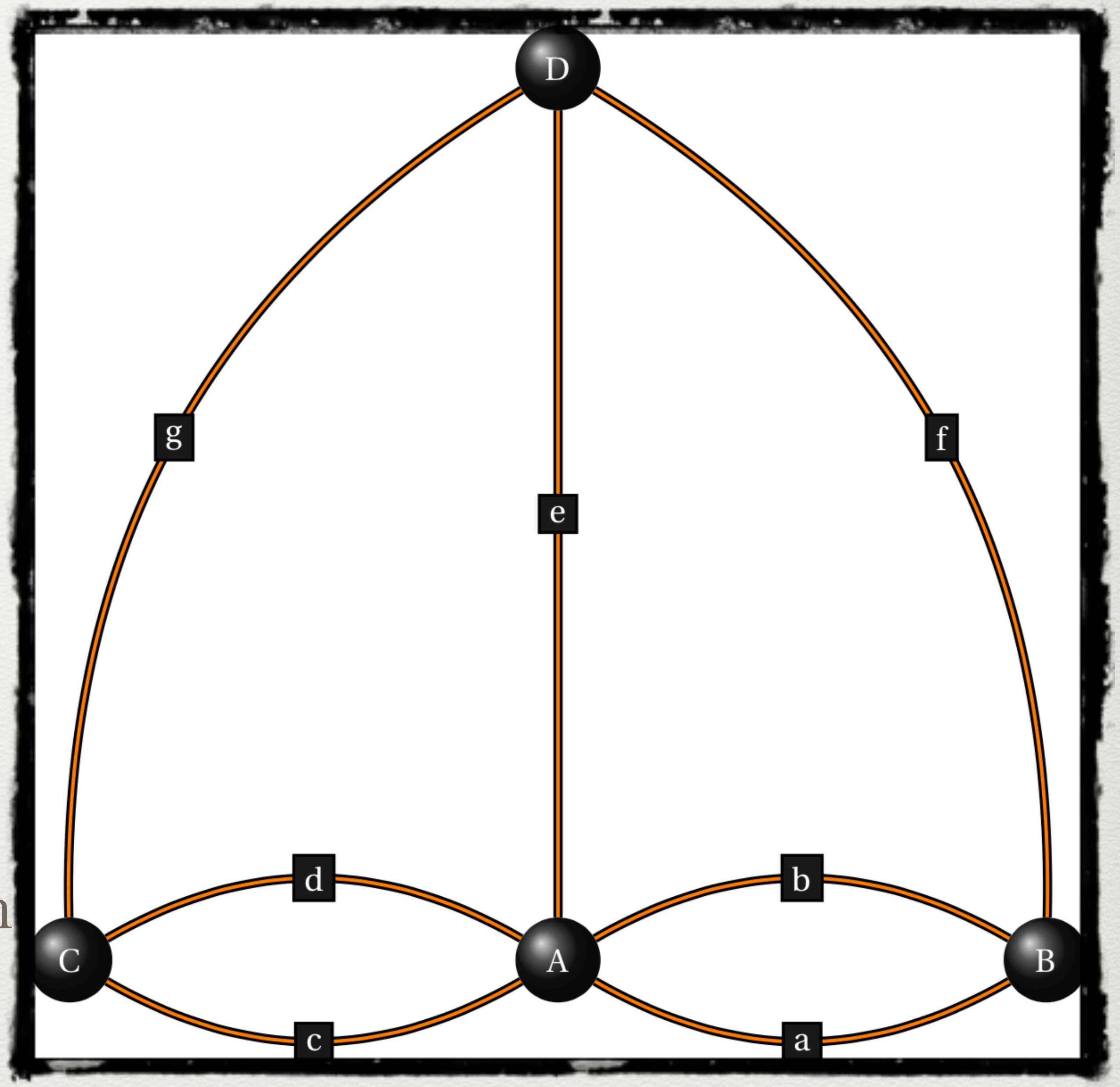
Euler's solution method

Euler's treatment of the Königsberg problem involved various major steps.

- Path: a sequence of vertices and edges,

  $n_0, e_1, n_1, e_2, n_2, \ldots n_{N-1}, e_N, n_N,$

  in which each edge e_i joins the nodes n_{i-1} and n_i $(1 \leq i \leq N)$.
- Problem reformulation: find a path which contains each edge of the graph once and only once (Eulerian path).



The graph analogy to the Königsberg problem.

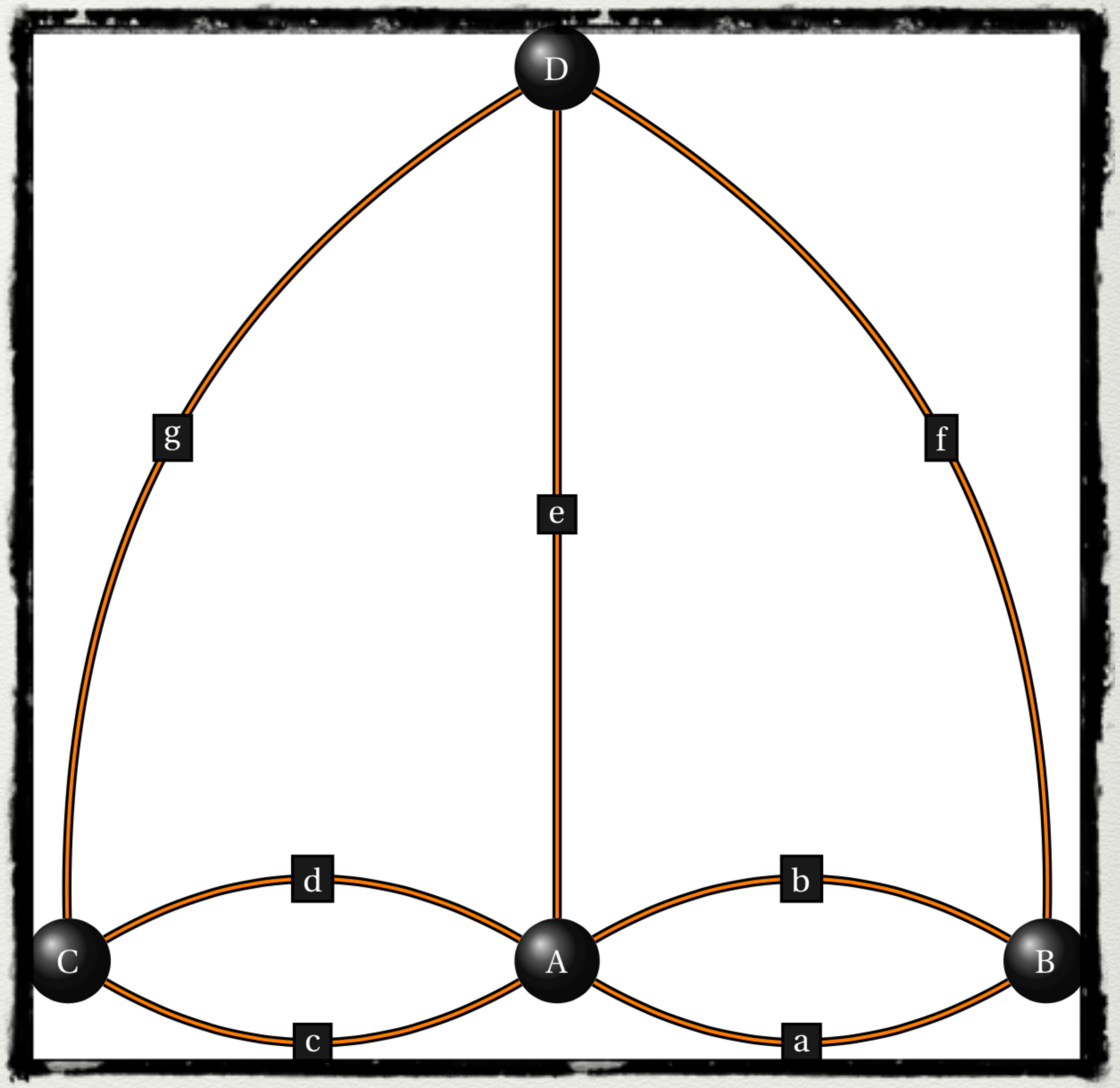
Euler's solution method

- Connected graph: for any two vertices *n* and *v* it is possible to find a path beginning at *n* and ending at *v*.

- Degree of a node *n*: The sum of the edges (or their weights) that meet at *n*.

Let $W \in \mathbb{R}^{N \times N}$, and considering unweighted graphs with $w_{ij} \in \{0,1\}$,

$$d_i = \sum_{j=1}^{N} w_{ij}\,.$$

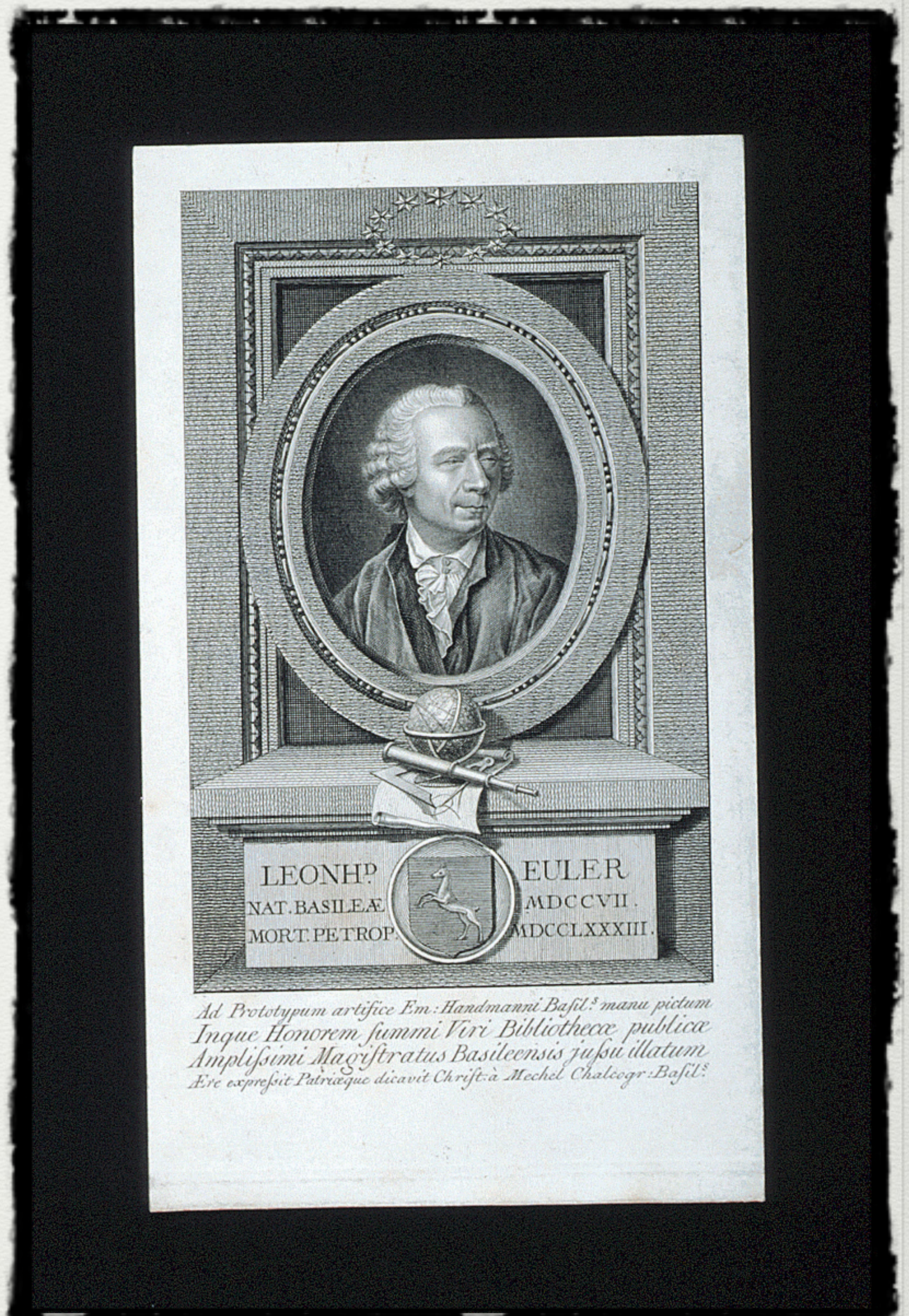


The graph analogy to the Königsberg problem.

Two main results

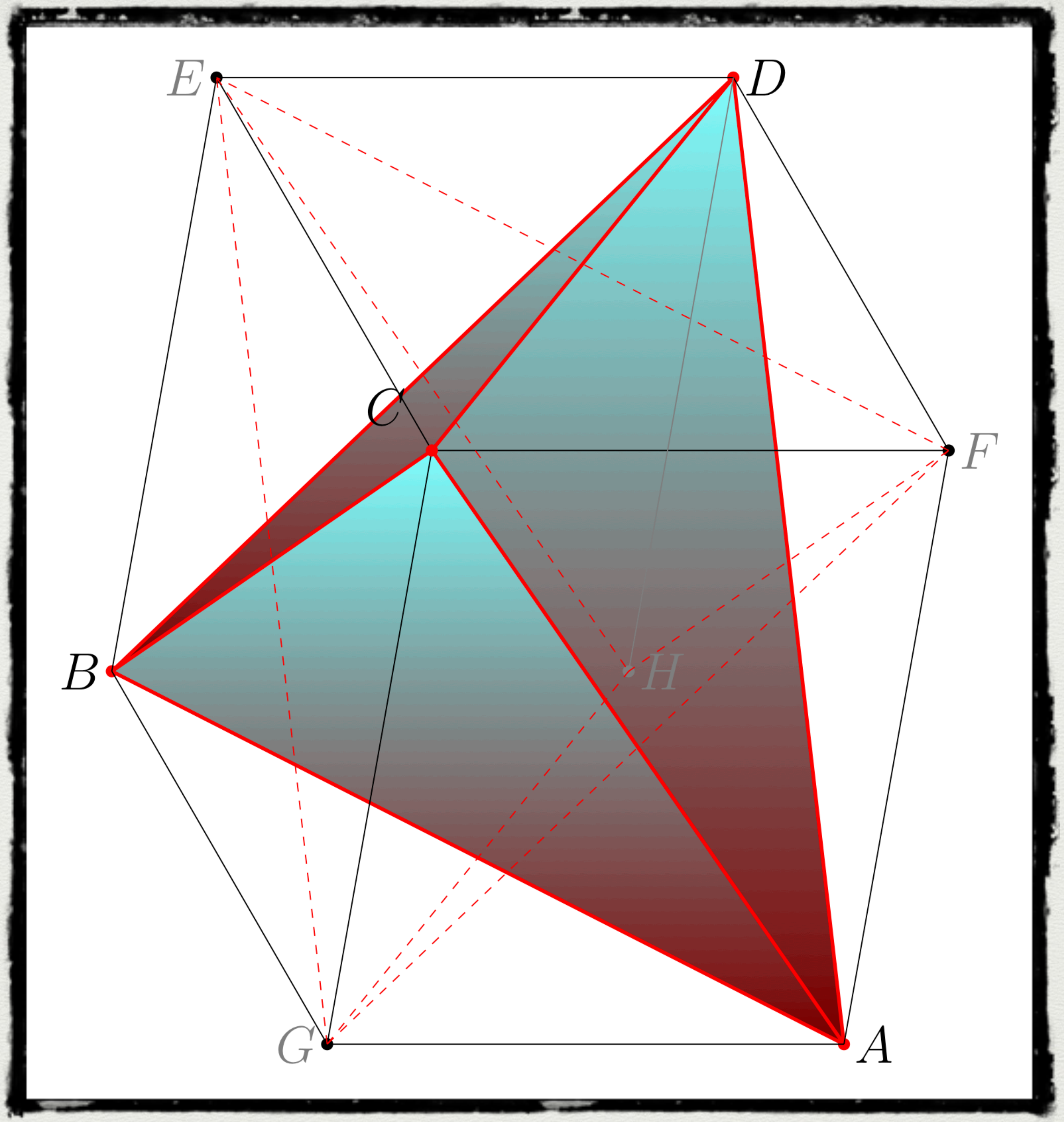
A. In any connected graph the number of vertices with odd degree must be even.

B. If a connected graph has more than two vertices of odd degree, then it cannot contain an Eulerian path.



Leonhard Euler (1707-1783)
Image credits: www.mhs.ox.ac.uk

Generalizing from his solution to this particular problem,
Euler derived the first results in **graph theory**.

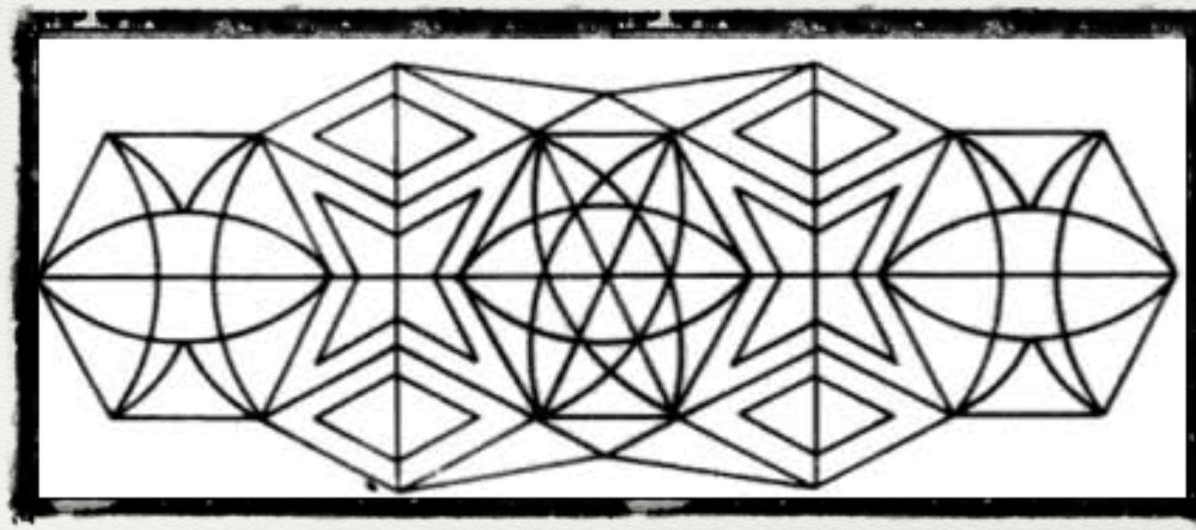


Graph representation of a tetrahedron inscribed in a parallelepiped.

4. Modern Graph Drawing

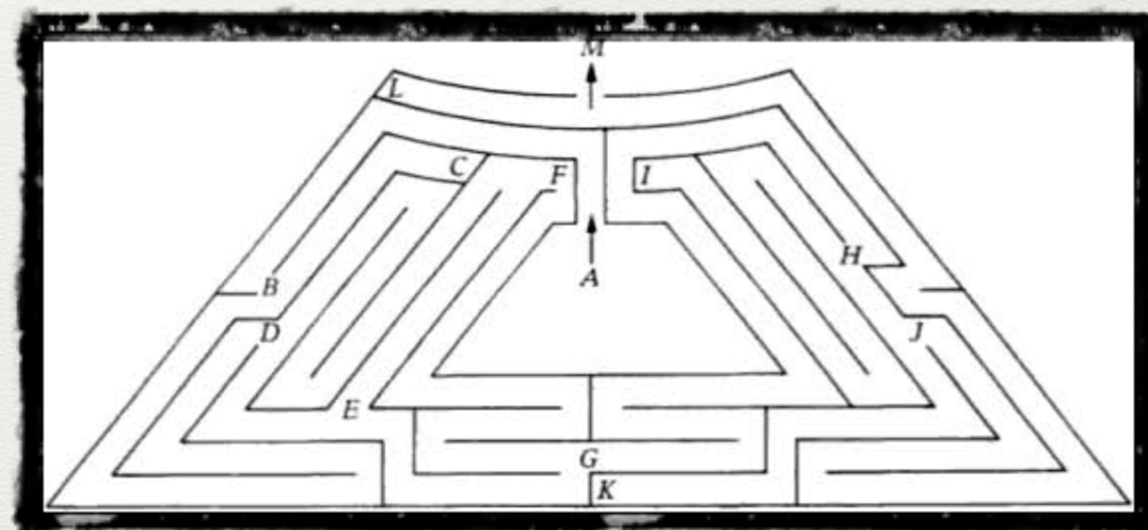
Paths

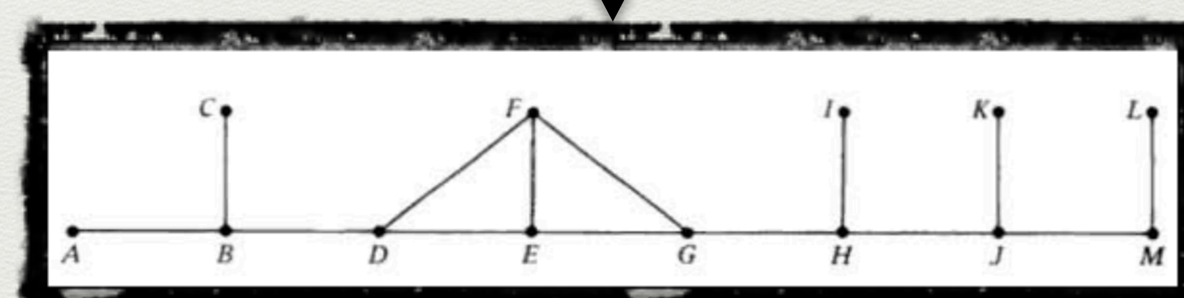
Diagram-tracing puzzles: Draw symmetric graphs that contain an Eulerian path.



J.B. Listing's single stroke drawing. There are only two nodes of odd degree, lying at the end of the horizontal line [6].

Mazes and labyrinths: How to escape? A path which starts from a given vertex (center), and ends at another given vertex (exit).





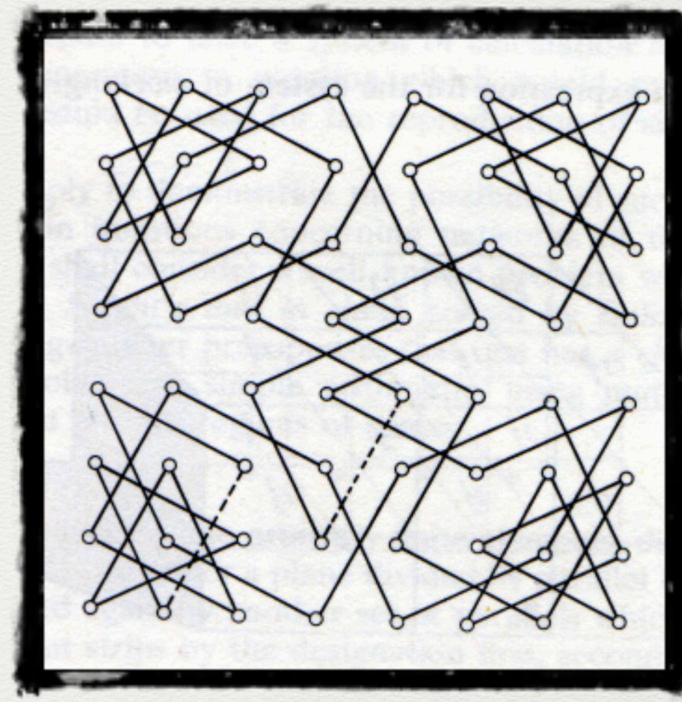
Algorithmic solutions

The corresponding graph of the Hampton Court maze in England [6].

Circuits

a path with the additional property that all of its vertices and all of its edges are distinct, besides the first and last vertices which must coincide.

The knight's tour: Find a sequence of moves on a chessboard, so that a knight may visit each square on the board just once, and finish on the same square as it began.

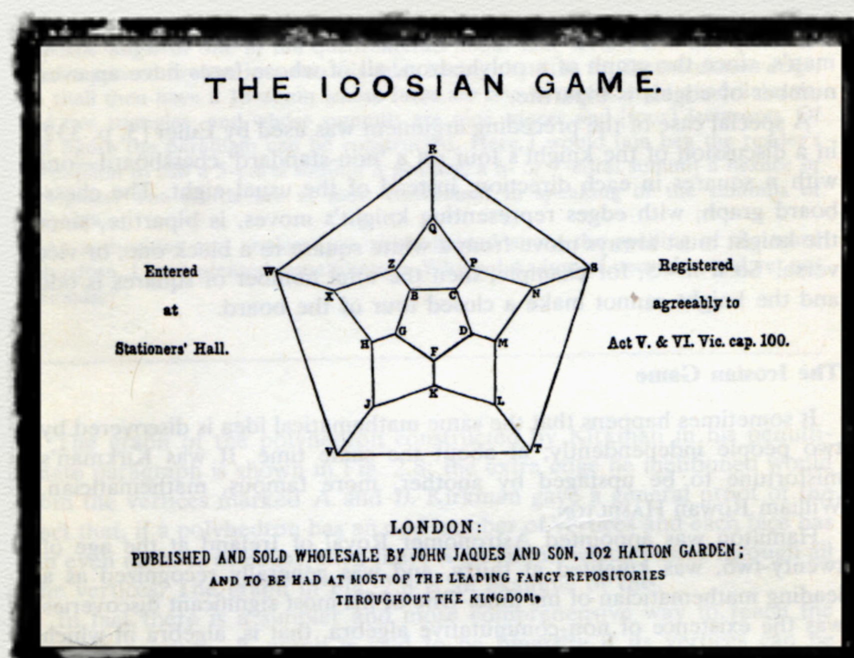


Vandermonde's 1771 graph drawing of a knight's tour. The nodes represent squares on a chessboard and edges legal moves [1].

The Icosian Calculus: Non-commutative algebra, i.e

$$xy \neq yx$$

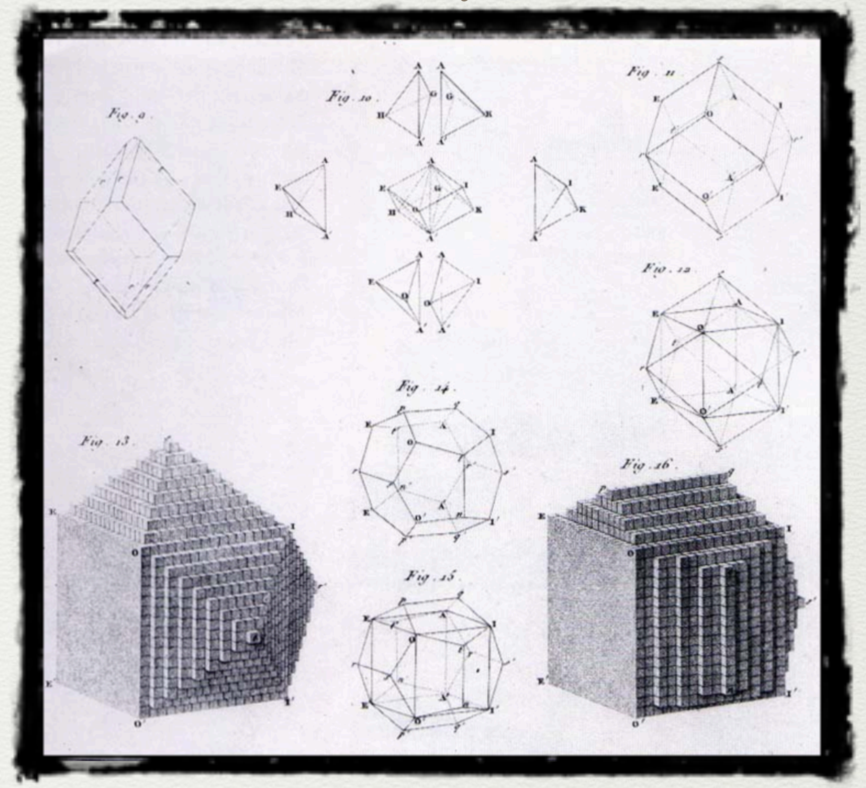
Sir W. R. Hamilton devised a game based on it, consisting of graph drawing. The objective was to find paths and circuits on the graph, satisfying certain condition (cover the board, finish cyclically etc.)



Hamilton's Icosian Game from 1857 [6].

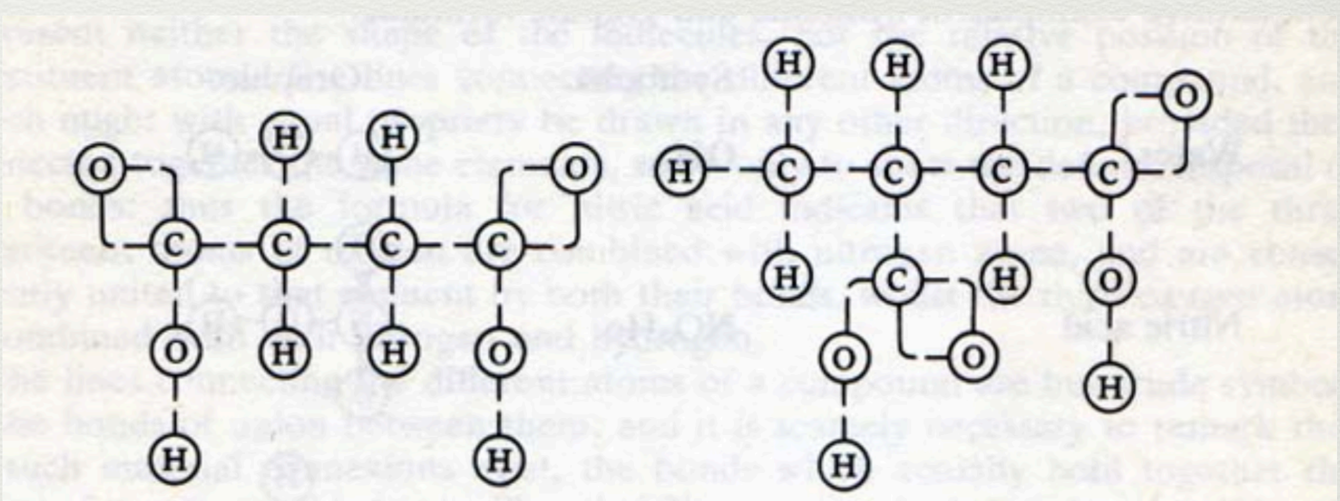
Crystallography & Chemistry

Renè Just Haüy (1743 - 1822): established the basic principles of crystallography. His abstract drawings of crystals represent a hybrid form of geometric and 3D drawing.

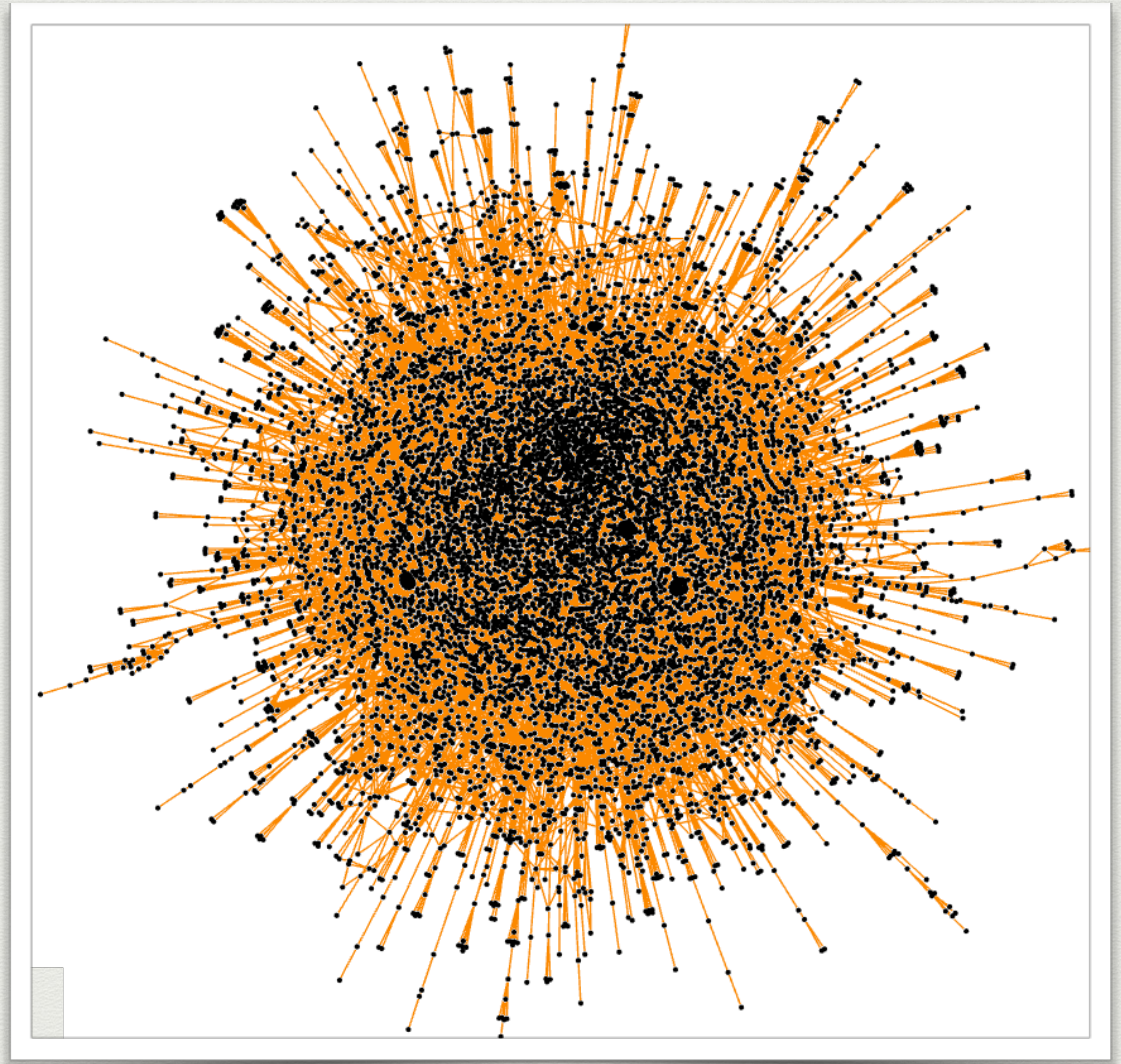


The geometry of crystal structures, 1784. The graph nodes correspond to corners or apexes of the physical crystal. Edges connect neighbouring nodes [1].

Alexander Crum Brown (1838 - 1922): introduced graphic notation in molecular chemistry. Each atom is shown separately, a letter enclosed by a circle, and bonds are marked by lines connecting the circles.



Brown's 1864 depictions of Succinic and Pyrotartatic acid. Modern drawing of molecules differ slightly from this style, as circles around atoms are usually neglected [6].

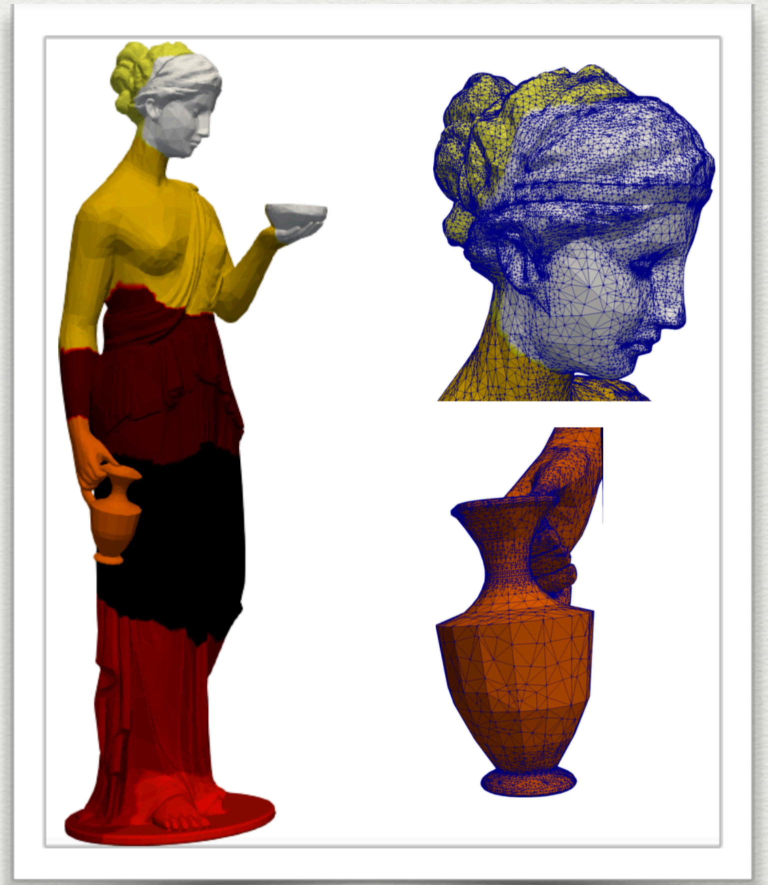


Solid physics co-citation network, 2012.

5. Applications of Graph Theory

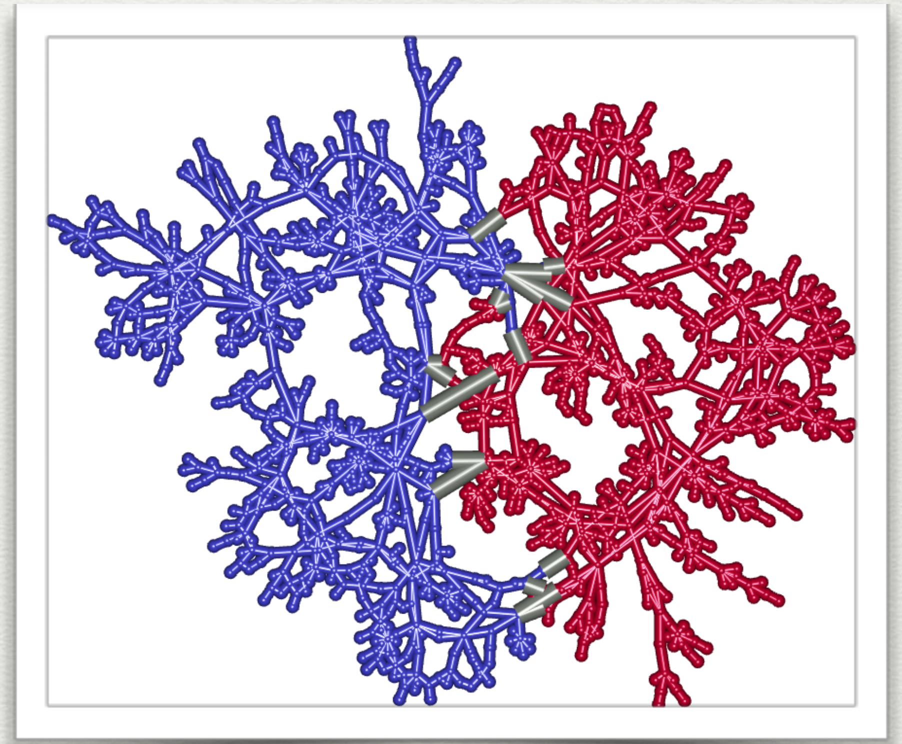
Modern applications

Mesh partitioning: map the subdomains of the mesh to processors for parallel computations.



Partitioning of finite element mesh.

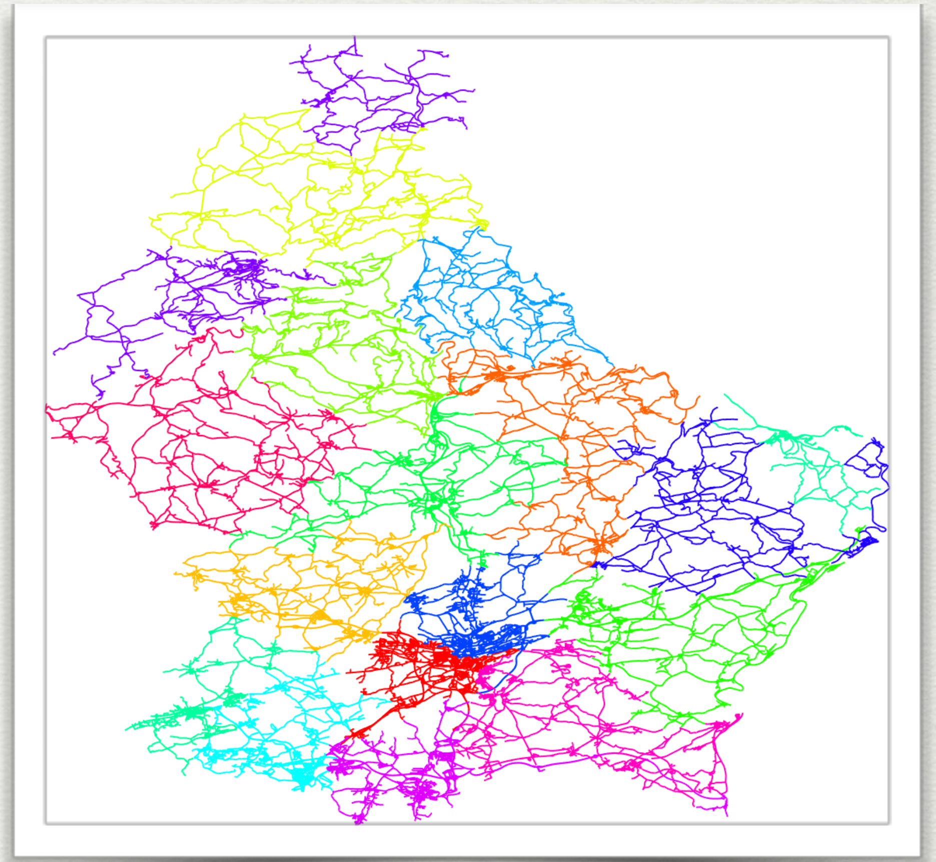
Power Grids: perform model reduction, defensive islanding, explore market power potential etc.



Bisection of part of the European power network.

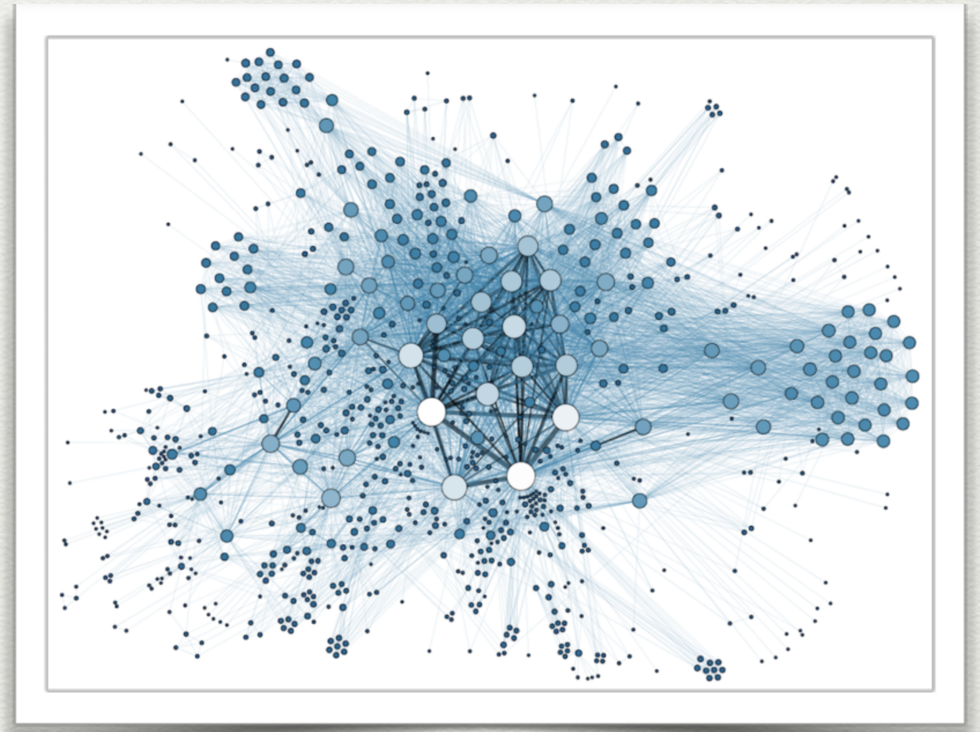
Modern applications

Road networks: speed up route planning, find shortest paths etc.



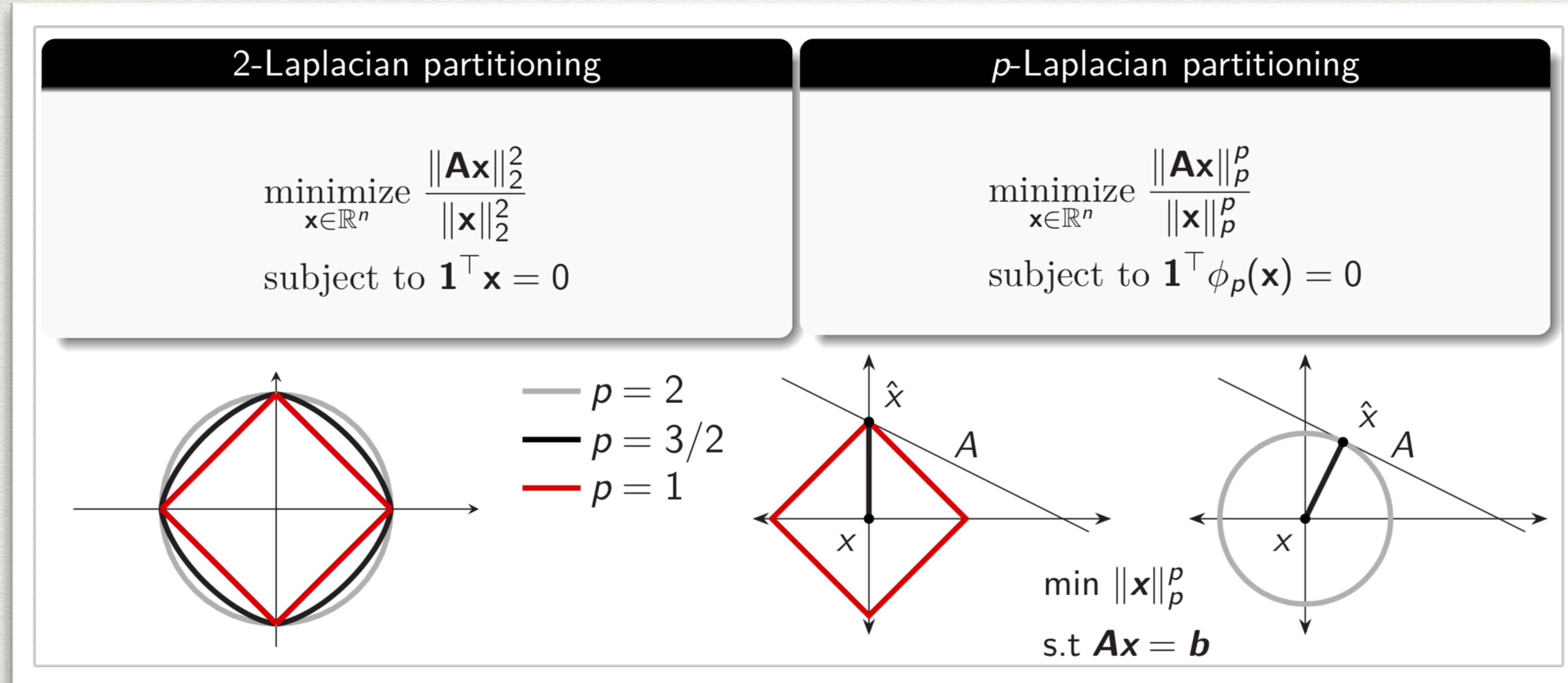
The road network of Luxembourg partitioned in 16 pieces

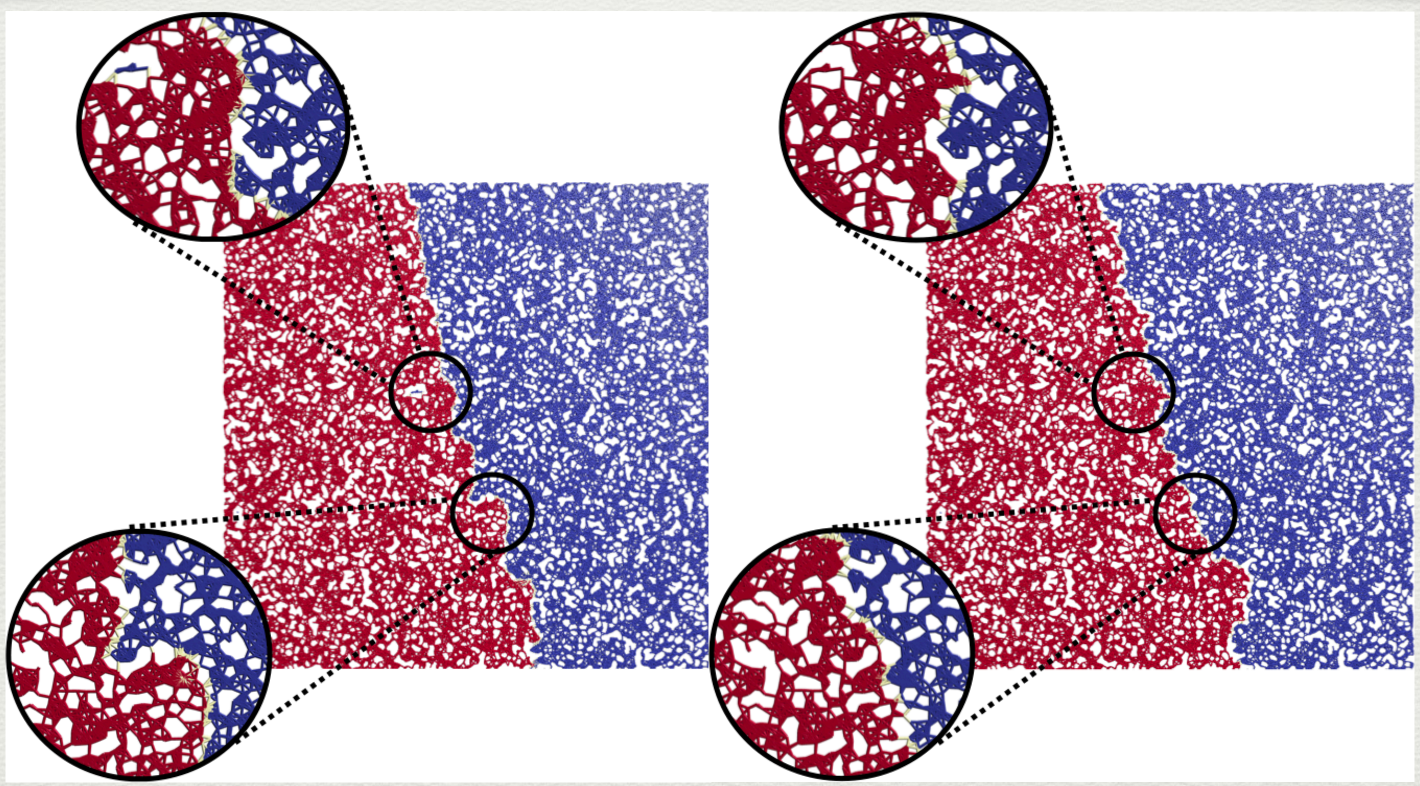
Social networks: identification of community structure, influential nodes in network etc.



Twitter users, network of 200.000, [Grandjean, 2016].

p-Laplacian partition refinement





Refining a random geometric graph rom the 10th DIMACS Implementation Challenge. Selected close-ups are illustrated. In the top ones the removal of isolated partition islands, present in the standard spectral bisection, is depicted. In the bottom ones, the removal of redundant cut edges is presented [2].

References

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